On the spectra of scale-free and small-world networks

Mircho Mirchev

This paper reviews the most commonly used models of complex systems and networks - Erdos-Renyi, Watts-Strogatz and Barabasi-Albert models. These models address different sets of properties and phenomena of real-life networks. Spectral analysis of the reviewed models has been made and also a dataset of Autonomous Systems (AS) links is been reviewed. Spectral analysis for the AS-links graph has been made and compared the spectra of the reviewed models. Based on this works, a new model can be developed, that better describes the unique characteristics of the biggest and most complex network - Internet.

Спектрален анализ на scale-free и small-world мрежи (Мирчо Й. Мирчев). *Настоящата разработка разглежда основните модели за изграждане на т.нар. комплексни системи и мрежи - модел на Ердьос-Рение, модел на Ватс-Строгац и модел на Барабаши-Алберт. Тези модели са създадени за описание на различни феномени наблюдавани в мрежите в реалния свят. Направен е спектрален анализ на представените модели, както и е изследвана базата от данни с връзките между автономните системи в Интернет. Също така е направена е съпоставка между спектралните характеристики на анализираните модели и мрежи. Въз основа на работата може да се разработи нов модел на изграждане на графи описващи една от най-големите мрежи в света - Интернет.*

Introduction

Networks are at the heart of some of the most revolutionary technologies and businesses of the 21st century, empowering everything from Google to Facebook, and Twitter. Understanding network structure and topological properties will give us the understanding of the complex systems that are powered by these networks.

The exploding interest in network science during the first decade of the 21st century is rooted in the discovery that despite the obvious diversity of complex systems, the structure and the evolution of the networks behind each system is driven by a common set of fundamental laws and principles. Therefore, notwithstanding the amazing differences in form, size, nature, age, and scope of real networks, most networks are driven by common organizing principles. Once we disregard the nature of the components and the precise nature of the interactions between them, the obtained networks are more similar than different from each other [1].

Graph theory has always been used to represent in mathematical way real-life scenarios. As each network can be represented as a graph, it has very wide application in network science – to calculate the shortest one or to search for critical edges (bridges) or critical nodes, called articulation points, evaluate some topological parameters or even to predict future development of the structure. Until recently most of these tasks were accomplished algorithmically.

In this paper we analyse the spectral properties of small-world and scale-free graphs, and then we compare these characteristics with some real-world network data [2], [3].

Definitions

As networks despite of their diversity are driven by a common set of laws and principles, there are models that can describe them. Small-world networks and scale-free networks are important complex network models with massive number of nodes and have been actively used to study the network topology of brain networks, social networks, and wireless networks. These models also have been applied to IoT networks to enhance synchronization, error tolerance, and more. However, due to interdisciplinary nature of the network science, with heavy emphasis on graph theory, it is not easy to study the various tools provided by complex network models. Therefore, in this paper, we attempt to analyse the spectral properties of small-world and scale-free network models as the spectrum of graphs gives a good evaluation on both local and global topological parameters of the network.

Small-world networks

Recently, the "small- world effect" or "six degrees of separation" principle, which was first discovered by the social psychologist Milgram in [4] and experimented in real-life as in [5]. Then it was systematically studied by Watts and Strogatz in [6]. In small-world networks, by randomly reconnecting a small number of links in a regular lattice network, the average path length is reduced significantly [7], [8]. Both random networks as per Erdos-Renyi [9] model (ER) and small-world (WS) networks have homogeneous network topology where the nodes have approximately the same number of links. So the degree distribution is Poisson with mean the average degree (close to *k,* in the Watts-Strogatz model, which is dependent of the network).

The small-world network model is based on starting with a regular lattice graph, where each node is connected to *k* of its neighbours and then each link is rewired with probability *p. (*Fig. 1*)*

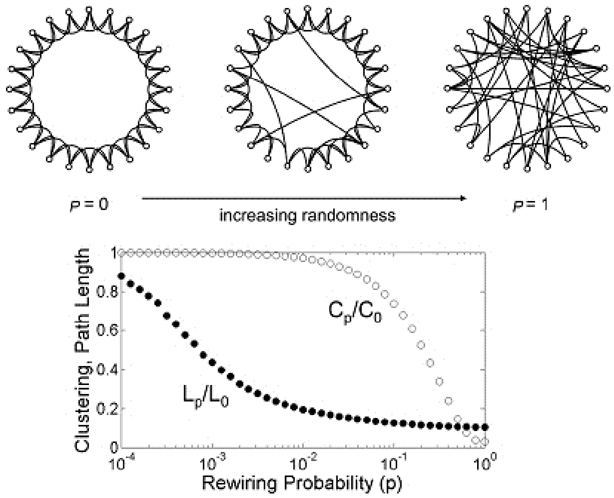

Fig. 1 - D. J. Watts and S. H. Strogatz, "Collective dynamics of small-world networks" Nature, vol. 393, no. 6684, p. 440, 1998.

Scale-free networks

Barabasi and Albert have discovered that many real world networks such as world wide web (WWW), social networks, and metabolic networks are not random with node connection or edge distribution approximated by Poisson distribution but have power-law distribution [10]–[13]. In contrast to the Poisson distribution, the power-law distribution has higher peaks and "fat" tails describing the existence of few nodes with massive links observed in real networks. In 1999, Barabasi and Albert proposed the scale-free network that has edge distribution equal to power-law nature. The two main features of the scale-free network are that it is an evolving network with incoming nodes and that these nodes are attached preferentially to the existing nodes with a large number of links Fig. 2

(1) $$P(k) = ck^{-\delta}, \quad \delta > 0, c > 0.$$

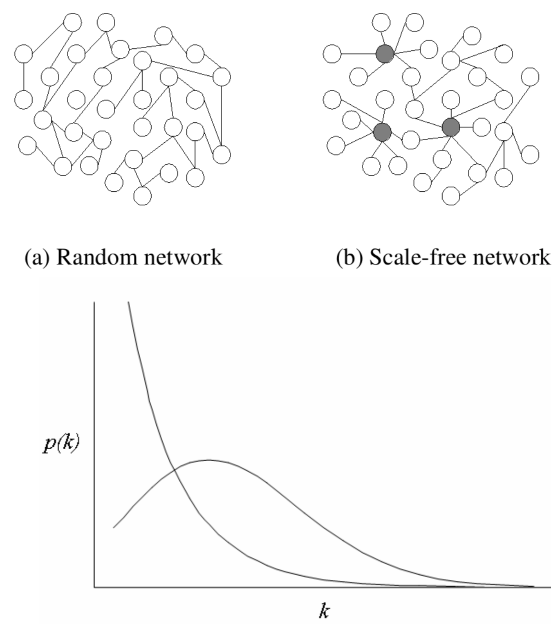

Fig. 2 - Random and Scale-free networks, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=646994

This model relies on the preferential attachment rule where some already highly connected vertices are likely to become even more connected than vertices with small degree. This model (BA model) generates a power-law degree distribution [14] which is observed in real-life networks – Fig.3.

Spectrum of graphs

The spectrum of finite graph G_c is by definition the spectrum of the adjacency matrix A, its set of eigenvalues together with their multiplicities. The Laplace spectrum of finite graph G_c is the spectrum of the Laplace matrix L [15]–[17].

Since A is real and symmetric, all its eigenvalues are real. Also, for each eigenvalue λ_n, its algebraic multiplicity coincides with its geometric multiplicity. Since A has zero diagonal, its trace *tr(A)*, and hence the sum of the eigenvalues is zero.

Similarly, L is real and symmetric, so that the Laplace spectrum is real. Moreover, L is positive semidefinite and singular, so we can denote the eigenvalues by:

(2) $$\lambda_n \geq \cdots \geq \lambda_2 \geq \lambda_1 = 0$$

The sum of these eigenvalues is *tr(L),* which is twice the number of edges of G_c. Finally, also $\mathcal{L}$ has real spectrum and nonnegative eigenvalues (but not necessarily singular) and *tr(ℒ)=tr(L)*.

In [18] the algebraic connectivity $a(G_C)$ of a (connected) graph is defined as the second smallest eigenvalue (λ_2) of the Laplacian matrix of a graph with *n* vertices.

This parameter is used as a generalized measure of "how well is the graph connected"[19]. It has values between *0* and *n* (a fully-connected graph K_n has *n*). This eigenvalue is greater than 0 if and only if G is a connected graph. This is a corollary to the fact that the number of times 0 appears as an eigenvalue in the eigenvector of the Laplacian is the number of connected components in the graph. Therefore, the farther λ_2 is from zero, the more difficult it is to separate a graph into independent components. However, the algebraic connectivity is equal to zero for all disconnected networks.

Fiedler vector is the eigenvector associated with the second smallest eigenvalue of the graph G – λ_2. These eigenvectors of the graph Laplacian have been explored extensively recently, mostly in [20]–[27]. The values of the Fiedler vector provide a good evaluation of the topology and the node significance in terms of global topology, e.g. to find densely connected clusters, to find poorly connected nodes, or to evaluate the global connectivity distribution of a network [28], [29].

Also, the differences between the values in the Fiedler vector can give an estimation of distance between nodes in graphs [18], [30], [31].

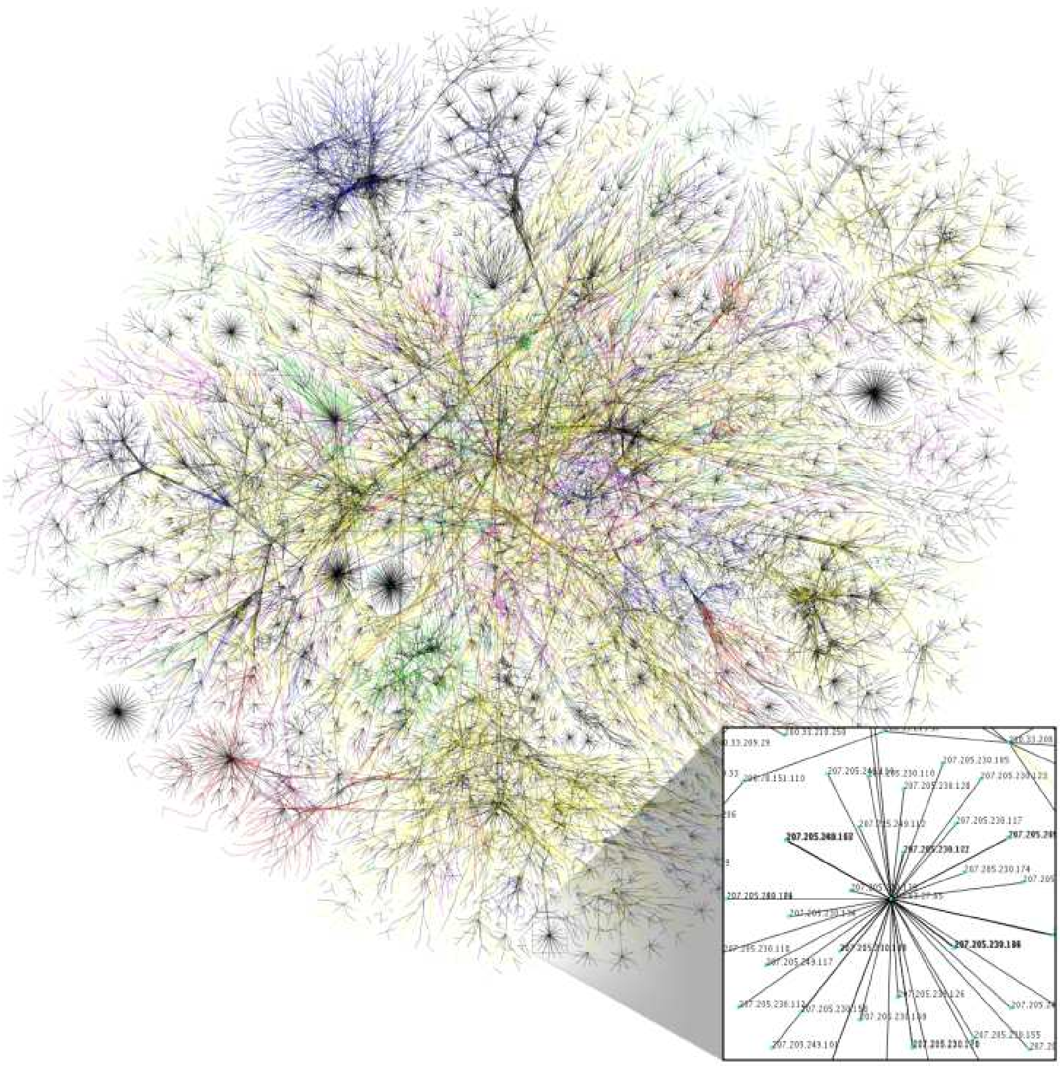

Fig. 3 - Internet map, by The Opte Project
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Spectra of Small-world and scale-free networks

The study of the characteristics of the spectrum of a random matrix by Wigner in 1955 [32] has increased interest in the behaviour of the spectral distributions of different matrices and graphs. Wigner introduced a law, known as Wigner's semicircle law [32]–[34]. According to this, the distribution of eigenvalues of a large real symmetric matrix, with elements taken from a probability distribution, follow a semicircle distribution.

In our research we'll investigate the spectra of WS and BA networks and compare them to the spectra of ER graphs. Since the spectrum of the normalized Laplacian matrix reflects global properties of the graph, and the spectrum of the adjacency matrix contains information about local properties of the graph, we'll investigate both, and also including the Laplacian (non-normalized) matrix eigenvalues distribution (other properties of Laplacian matrix are dependent on the size of the network).

While it is known that the BA network follows the power-law in its degree distribution, further structural properties are not so well known. When $m = 1$, the BA network forms a tree structure without forming any loop, but for $m > 1$, loops are formed, and network topology becomes much complicated. So it would be interesting to investigate the spectrum of the BA network, because generally the spectrum of a random graph and corresponding eigenvectors are closely related to topological features of the random graph. In this section, we study the spectrum and the corresponding eigenvectors of the adjacency matrix of BA networks, comparing spectral properties with structural features.

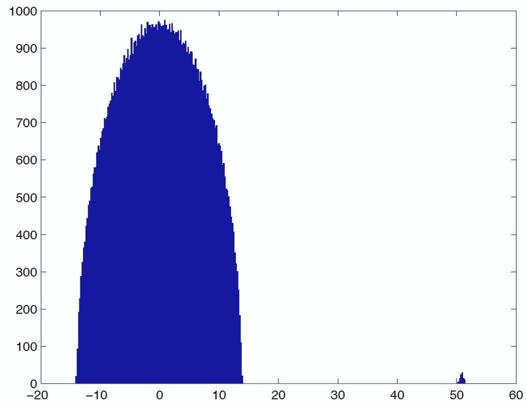
We have done spectral analysis of the following types of graphs:

- Random *G(n,p)* graphs (ER) – based on the Erdos-Renyi model, where *n* is the size of the graphs, and *p* is the probability for a node to create an edge.
- Small-world *G(n,k,p)* graphs (WS) – based on the Watts-Strogatz model, where *n* is the size of the graphs, *k* - the initial neighbours that each node is connected to, and *p* is the rewiring probability
- Scale-free *G(n,m)* graphs (BA) – based on the Barabasi-Albert model, where *n* is the size of the graphs, and *m* is the number of edges that each new node creates using the Preferential attachment model.

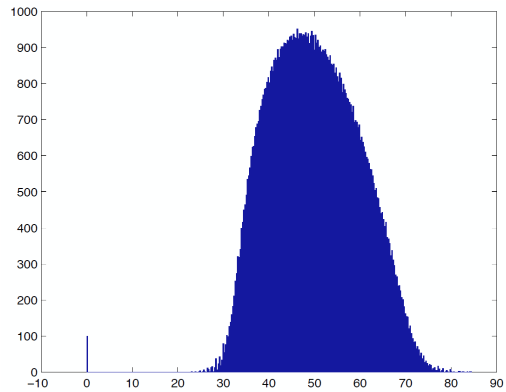
For spectra visualizations we use histogram plots of the Eigen values of the respective matrices – sum over 100 realizations, with 317 bins within the interval $[\lambda_{min}; \lambda_{max}]$. The number of bins follows a very rough pattern of ten times the square root of the size of the network, which in overall gives best representation. Size of all networks is 1000 nodes.

On Fig. 4 are shown the plots of spectra – histograms of Eigen values – sum over 100 realizations of ER graphs with *p=0.05* of (a) Adjacency Matrix; (b) Laplacian Matrix; (c) Normalized Laplacian matrix

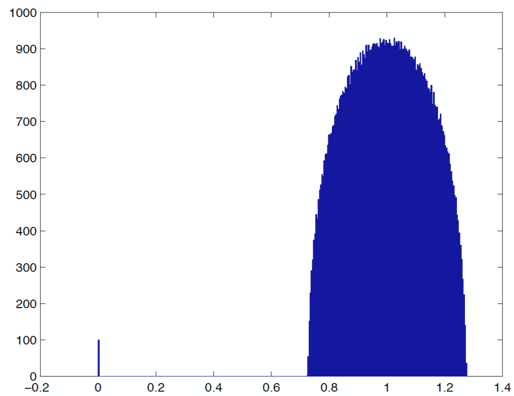
As expected the spectra are per Wigner's law - semi circle. Later we'll compare these spectra to the spectra of WS and BA graphs, and in next section to the spectra of the Internet AS links graph.



(a) Adjacency matrix;



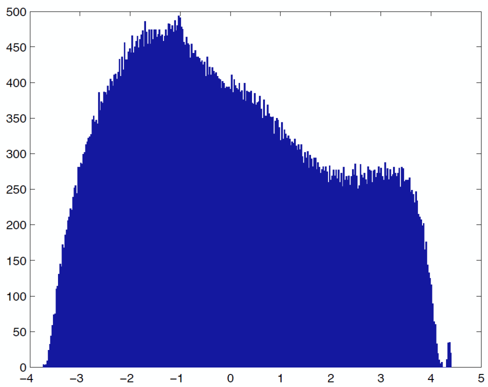
(b) Laplacian matrix;



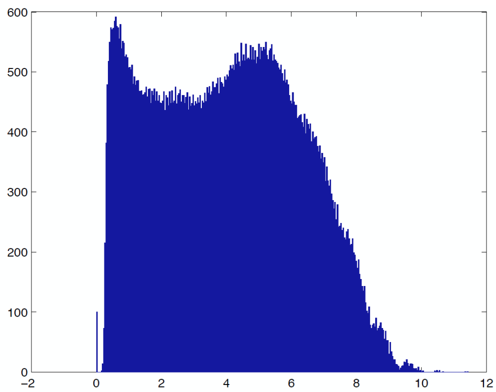
(c) Normalized Laplacian matrix

Fig. 4 - Spectrum of Erdos-Renyi (ER) random graph.

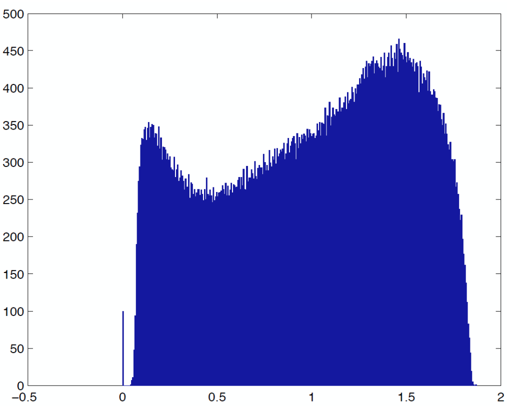
On Fig. 5 are shown the spectra of WS graphs (sum over 100 realizations) with starting lattice of average degree of 4 and rewiring probability *p=0.3*. For clarity in comparison, number of bins is the same (317) and size of networks is again 1000. (a) Adjacency Matrix; (b) Laplacian Matrix; (c) Normalized Laplacian matrix



(a) Adjacency matrix;



(b) Laplacian matrix;

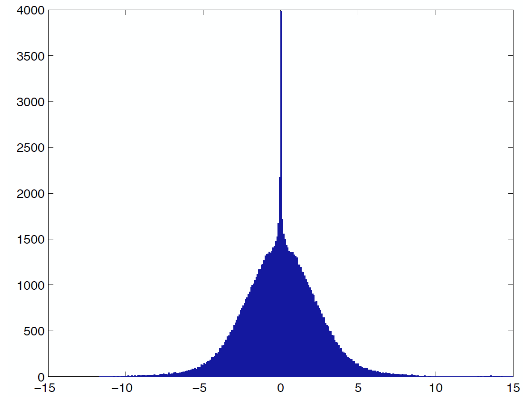


(c) Normalized Laplacian matrix

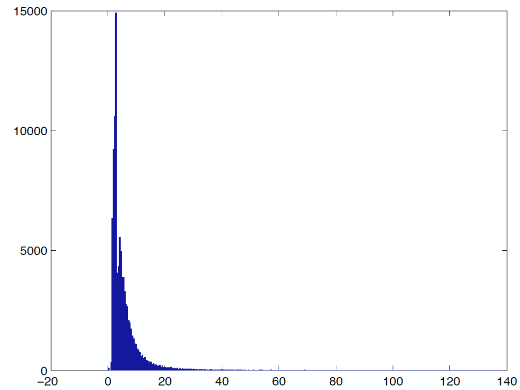
Fig. 5 - Spectrum of Small-Network (WS) graphs.

The spectra of WS graphs are quite different from the ER graphs. It is seen that the spectra are a lot more distributed.

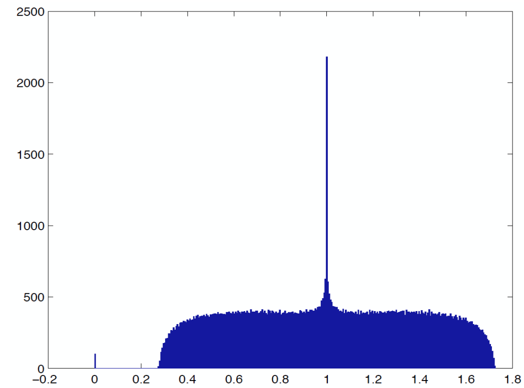
On Fig. 6, Fig. 7 and Fig. 8 are shown the spectra of BA graphs (sum over 100 realizations). The BA graphs are with different parameters, so that the changes in spectra can be evaluated. For clarity in comparison both with ER, WS and other BA graphs shown here, number of bins is the same (317) and size of networks is again 1000.



(a) Adjacency matrix;

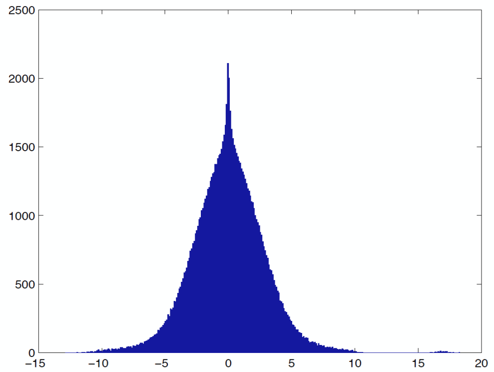


(b) Laplacian matrix;

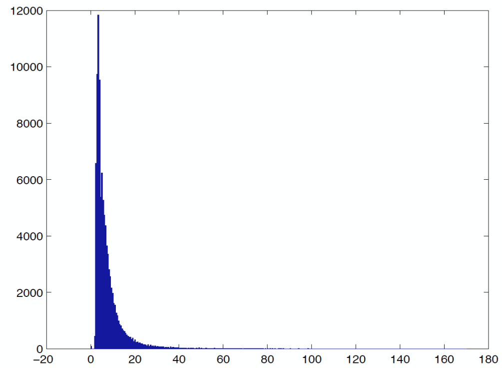


(c) Normalized Laplacian matrix

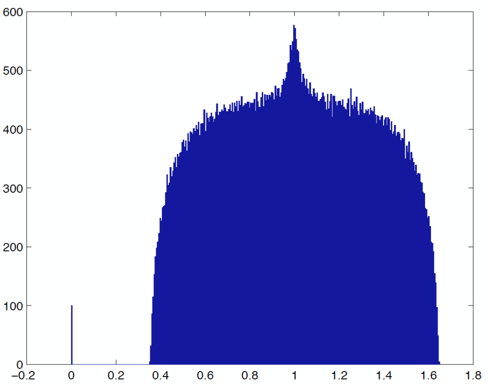
Fig. 6 - Spectrum of Scale-free (BA) graphs with m=3



(a) Adjacency matrix;

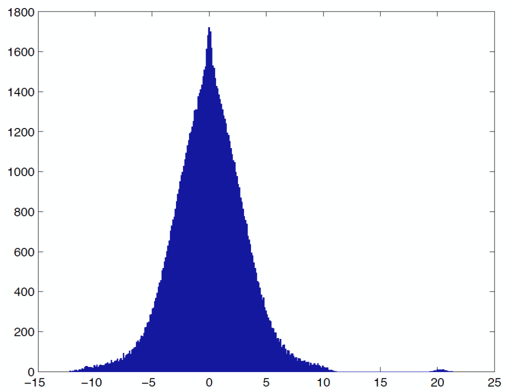


(b) Laplacian matrix;

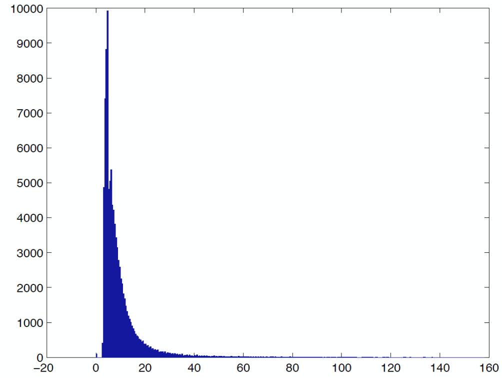


(c) Normalized Laplacian matrix

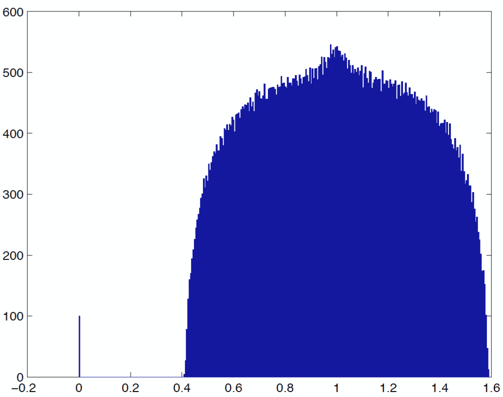
Fig. 7 - Spectrum of Scale-free (BA) graphs with m=4.



(a) Adjacency matrix;



(b) Laplacian matrix;



(c) Normalized Laplacian matrix

Fig. 8 - Spectrum of Scale-free (BA) graphs with m=5.

It is clear from the pictures above that the *m* parameter only scales the spectra, while the shape is the same.

Spectra of Internet AS links

In this section we take real data from Caida Internet AS relationship [2] and make spectral analysis on the graph presented. We then compare these spectra to spectra of ER, WS and BA graphs, which parameters are derived from the AS relationship graph as follows:

- Probability p for ER graphs is taken from the density of the AS graph which in the current dataset it is $\cong 6 \cdot 10^{-4}$
- Initial neighbours to connect in WS graph *m*, and degree of new nodes in BA graph *m* - both are the same and derived from the average node degree in the AS links graph as a rough estimate.

First we can confirm the power-law distribution of the degrees of nodes on Fig. 9. From this we can derive that the closest model to the AS links graph is the BA model.

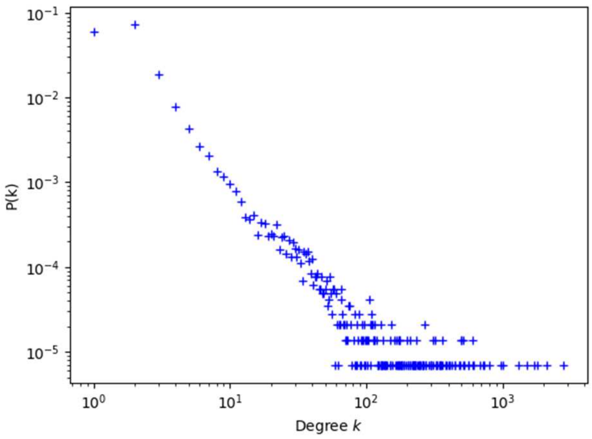
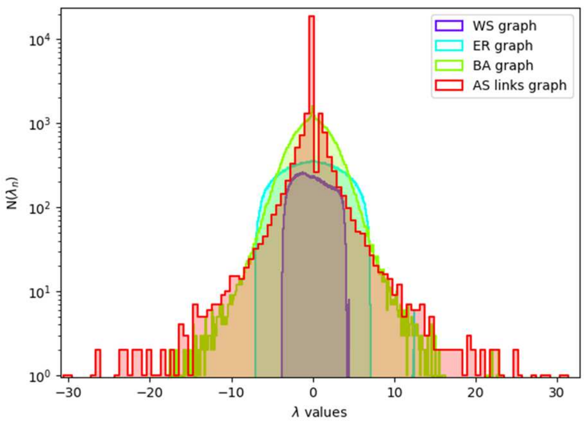
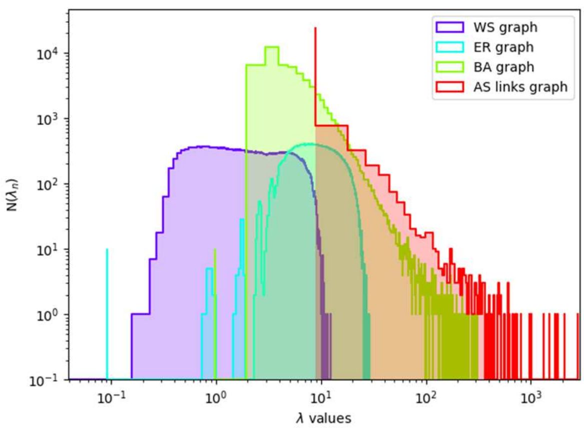


Fig. 9 - Degree distribution of AS links graph.

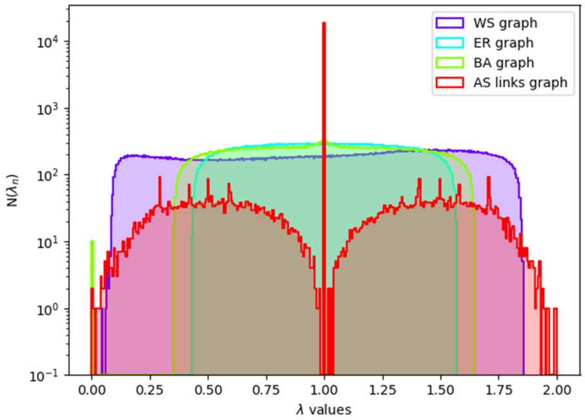
In Fig. 10 is shown an overlay of the spectra of compared graphs (ER, WS, BA and AS-Links). From the picture is seen that the AS-Links graph spectra have shape that is similar both the WS and BA graphs. This is expected as Internet is considered as a network with both small average shortest-path lengths, as in small-world networks and with exponential node degree as in scale-free networks.



(a) Adjacency matrix;



(b) Laplacian matrix;



(c) Normalized Laplacian matrix

Fig. 10 - Spectra of ER, WS, BA and AS-links graphs

Conclusions and future works

In this paper we have reviewed the spectra of the most widely used today network models that can approximate the real-world networks, including the Internet. Then we compared the spectra of the Adjacency matrix, the Laplacian matrix and the normalized Laplacian matrices for Erdos-Renyi (ER) random graphs, for Watts-Strogatz (WS) graphs and Barabasi-Albert (BA) graphs. We derived the conclusion that for WS and BA graphs, the initial parameter doesn't influence the shape of spectra.

Then we made spectral analysis of a dataset of real network - the Caida dataset of AS Relationships as for Sept 2017 and compared these spectra to the spectra of ER, WS and BA graphs with similar densities.

Future works on the topic are to propose models that incorporate properties from both WS and BA model, as the spectra of Internet incorporates elements from spectra of both models. Such models can better describe Internet and other real-life networks that have properties from several models and can be used for further topology analysis of complex networks and systems.

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